ALGEBRAIC GEOMETRY - MIDSEMESTER EXAMINATION

Attempt all questions - Total Marks 30 - 10 a.m. to 1 p.m., 21st Feb 2011

The field k is algebraically closed in all questions. You may quote results proved in class. I have asked you to be brief in two questions - Question 2 and the first part of Question 7. You do not have to show all calculations and quote all relevant results in these two questions. For example, if you state a certain ideal which arises in your problem as prime, I will accept it if it is indeed so. Remember this holds only for these two questions (Qn 2 and first part of Qn 7).

(1) Prove or disprove: $V_1 = V(Y^2 - X) \subset \mathbb{A}_k^2$ is isomorphic to $V_2 = V(XY - 1) \subset \mathbb{A}_k^2$. (**3 marks**)

(2) Find the irreducible components of $V = V(XZ - X, X^2 - YZ) \subset \mathbb{A}_k^3$, and also describe the ideal of each component (be brief with your answer). (**3 marks**)

(3) Let V denote the affine variety $V = V(XW - YZ) \subset \mathbb{A}_k^4$. Let $\Gamma(V) = \frac{k[X,Y,Z,W]}{(XW-YZ)}$ be the affine coordinate ring of V, and let $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ be the residues of X, Y, Z, W (respectively) in $\Gamma(V)$. Let $f \in k(V)$ be a rational function on V defined as $f = \bar{X}/\bar{Y} = \bar{Z}/\bar{W}$. Show that it is impossible to write $f = g/h$, where $g, h \in \Gamma(V)$, and $h(P) \neq 0$ for every P where f is defined. (**5 marks**)

(4) Let L be a line in $\mathbb{A}_k^2$ and $F \in k[X, Y]$ define a curve in $\mathbb{A}_k^2$. Prove that L is tangent to the curve F at a point P if and only if $I(P, F \cap L) > m_P(F)$. (**3 marks**)

(5) Let $F, G, H \in k[X, Y]$ define curves in $\mathbb{A}_k^2$. If P is a simple point on F, then show that $I(P, F \cap (G + H)) \geq \min(I(P, F \cap G), I(P, F \cap H))$. Give an example to show that this may be false if P is not a simple point on F (hint: for the example choose F with atleast two distinct tangents at P). (**3+4 marks**)

(6) Let $F \in k[X, Y]$ define a curve in $\mathbb{A}_k^2$. Suppose P is a double point on the curve F, and suppose F has only one tangent L at P. Show that $I(P, F \cap L) \geq 3$.(**3 marks**)

(7) Let $V = V(Y - X^2, Z - X^3) \subset \mathbb{A}_k^3$. Prove that $I(V) = (Y - X^2, Z - X^3)$ (be brief with your answer). Show that $ZW - XY \in I(V)^* \subset k[X, Y, Z, W]$, but $ZW - XY \notin ((Y - X^2)^*, (Z - X^3)^*)$ (F^* is the homogenization of $F \in k[X, Y, Z]$ with respect to W, and $I(V)^*$ is the ideal generated by all F^* for $F \in I(V)$). (**2+4 marks**)